TTIC 31260 - Algorithmic Game Theory (Winter 2026)

Lecturer: Avrim Blum Lecture 12: February 18, 2026

Mechanism Design IV: Combinatorial Auctions

1 Overview

In this lecture we consider the problem of maximizing social welfare in combinatorial auctions. We will consider the setting with n buyers and m items (and we just have one copy of each item). Buyer i has valuation function v_i and we assume each buyer is only interested in what he/she gets (v_i only depends on buyer i's allocation). We already know that the VCG mechanism maximizes social welfare, but this can be computationally hard. We will see today that we can approximately maximize social welfare in the case of buyers with subadditive valuations using a simpler scheme in which we set prices to each item.

2 Winner Determination and Valuation Functions

First, let us forget about incentives and prices, and only worry about the algorithmic problem of "winner determination": given a set of v_i's, solving for an allocation of the items so that the social welfare is maximized. There are several classes of valuation functions that we can consider:

1. Additive valuation functions: $\forall i,\ v_i(S) = \sum_{x \in S} v_i(\{x\})$. We can maximize social welfare simply by giving each item to the buyer who values it the most.
2. Unit demand valuation functions: $\forall i,\ v_i(S) = \max_{x \in S} v_i(\{x\})$. In this case we can reduce the winner determination problem to a bipartite maximum-weight matching problem: put the buyers on one side and the items on the other side, and add an edge of weight $v_i(\{x\})$ between buyer i and item x).
3. Single minded valuation functions: $\forall i, \exists S_i$ with some valuation $v_i(S_i)$ such that for all sets S, if $S \supseteq S_i$ then $v_i(S) = v_i(S_i)$ and if $S \not\supseteq S_i$ then $v_i(S) = 0$. In other words each buyer has a single set they want. In this case maximizing social welfare becomes the set packing problem.
4. Subadditive valuation functions: $\forall i, S, T\ v_i(S \cup T) \leq v_i(S) + v_i(T)$.
5. Submodular valuation functions: $\forall i, S, T\ v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$.

   Another way to look at this is to rewrite it as: $v_i(S \cup T) - v_i(S) \leq v_i(T) - v_i(S \cap T)$. In particular, for any three disjoint sets A, B, C (let $S = A \cup B$ and let $T = B \cup C$), the marginal value of C given that you already have B is greater than or equal to the marginal value of C given than you already have $A \cup B$.

It turns out that maximizing social welfare is NP-hard in the last three cases.

3 Setting Prices

We now show how we can achieve fairly high social welfare for buyers with *subadditive valuation functions*. This material comes from the paper:

> Maria-Florina Balcan, Avrim Blum, and Yishay Mansour, "Item Pricing for Revenue Maximization." *Proc. 9th ACM Conference on Electronic Commerce*, pp. 50-59. 2008.

The mechanism will be very simple. We will put the *same* price on each item, have the buyers enter in order from $1, ..., n$ and purchase whatever they want in our "dollar store" from among the items remaining. We could implement this as a direct-revelation mechanism by having buyers submit their valuations up front, and then running this procedure virtually. The mechanism is incentive compatible because it is selecting the utility maximizing bundle for you according to the valuation you provide, and the valuation submitted does not impact the prices or the items available.

Definition 1 *A set S_i is supported at price p for buyer i if for all $W \subseteq S_i$, $v_i(W) \geq p|W|$.*

Claim 1 *Suppose v_i is subadditive, buyer i is shown a set T_i with each item at price p, and the buyer buys $S_i \subseteq T_i$ (i.e. $S_i = \arg\max_{S \subseteq T_i} v_i(S) - p|S|$). Then S_i is supported at price p.*

Proof: Suppose for a contradiction that the claim does not hold. Then there exists $W \subseteq S_i$ such that $v_i(W) < p|W|$. Then since v_i is subadditive,

$$v_i(S_i) \leq v_i(S_i \setminus W) + v_i(W)$$
$$v_i(S_i) < v_i(S_i \setminus W) + p|W|$$
$$v_i(S_i) - p|S_i| < v_i(S_i \setminus W) - p|S_i \setminus W|.$$

Thus $S_i \setminus W$ is preferred to S_i and this is a contradiction. ■

Suppose $T_1, \ldots, T_n$ is the social welfare maximizing allocation (buyer i gets set T_i), and assume $1 \leq \max_S v_i(S) \leq H$ for some maximum value H. Imagine we pick a random price p among $\{H, H/2, H/4, \ldots, \frac{1}{4n}\}$, assign price p to all items, and somehow manage to only allow each buyer i to select from T_i. Let $L_{i,p}$ be the set buyer i would choose. In the last lecture we showed that

$$E\left[p \cdot |L_{i,p}|\right] = \Omega\left(\frac{v_i(T_i)}{\log(mH)}\right).$$

This leads us to the following mechanism and analysis. Pick p at random from $\{H, H/2, H/4, \ldots, \frac{1}{4n}\}$ and set the price of each item to $p/2$. Have the buyers enter the store in order $1, \ldots, n$ and purchase whatever they want from the items remaining. Let S_i be the set that buyer i purchases, and let $W_i = L_{i,p} \setminus (\cup_{j=1}^{i-1} S_j)$. That is, W_i is the set of items still in the store when buyer i arrives from the set $L_{i,p}$ that the buyer would have purchased in the above thought experiment. Since $L_{i,p}$ is supported at price p, we have

$$\begin{aligned}
p|W_i| &\leq v_i(W_i) \\
\frac{p}{2}|W_i| &\leq v_i(W_i) - \frac{p}{2}|W_i| \\
\frac{p}{2}|W_i| &\leq v_i(S_i) - \frac{p}{2}|S_i| \\
& \qquad \text{(buyer } i \text{ could have chosen } W_i \text{ but chose } S_i\text{)} \\
v_i(S_i) &\geq \frac{p}{2}|W_i| + \frac{p}{2}|S_i|
\end{aligned}$$

Summing over all buyers, we get

$$\begin{aligned}
\sum_i v_i(S_i) &\geq \frac{p}{2}\sum_i (|S_i| + |W_i|) \\
&\geq \frac{p}{2}\sum_i |L_{i,p}|.
\end{aligned}$$

The last inequality follows from the fact that every item in $L_{i,p}$ is either still there when buyer i comes in, in which case it is counted in $|W_i|$, or is bought by some buyer $j < i$, in which case it is counted in $|S_j|$. Moreover, the sets $L_{i,p}$ are disjoint, so we are not double-counting any items. Finally, we can plug in our revenue guarantee from last lecture to get:

$$E\left[\frac{p}{2}\sum_i |L_{i,p}|\right] = \Omega\left(\frac{\text{max social welfare}}{2\log(mH)}\right).$$

4 Walrasian Equilibrium (Market Equilibrium)

We'll now consider item-pricings where different items may have different prices.

Definition 2 *Consider some pricing $p_1, \ldots, p_m$ on items. We define the demand set D_i for bidder i to be* $\arg\max_S v_i(S) - p(S)$ *where $p(S) = \sum_{j \in S} p_j$.*

Definition 3 *A Walrasian equilibrium is a set of prices $p_1, \ldots, p_m$ and an allocation $S_1, \ldots, S_n$ (set S_i allocated to bidder i and all sets are disjoint) such that S_i is a demand set for buyer i. Furthermore, any unallocated item has price zero.*

Note that if there are no ties in defining the demand sets (each buyer has a unique prefered set) then a Walrasian equilibrium means that all the buyers can come in at the same time and buy what they want, and there will be no contention. Moreover, even if there are ties, we can still assign sets to buyers so that they each are getting a favorite set at these prices.

Unfortunately, Walrasian equilibria do not always exist. For example, suppose there are two buyers and one thousand items. The first buyer is single minded and wants everything and values the "grand bundle" at 1000. The second buyer has unit demand and has value 10 on any one item. Suppose the total price of the items is less than 1000, then there will be contention (we cannot give each buyer their demand set), as the first buyer will want everything and the second buyer will want at least one item. If the total price is greater than 1000, then the first player does not want anything, and the second player wants at most one thing costing at most 10. Thus there are unallocated item with nonzero prices.

Theorem 1 *If a Walrasian Equilibrium exists, then the allocation maximizes social welfare.*

Proof: Here is a LP-relaxation of the social welfare maximization problem. Let x_{iS} indicate whether we allocate set S to buyer i.

$$\begin{aligned}
\max \sum_i \sum_S & x_{iS} v_i(S), & \\
\text{s.t.} \quad \sum_{S \ni j} \sum_i & x_{iS} \leq 1 & \forall \text{ items } j \ (j \text{ goes to at most 1 buyer}) \\
\sum_S & x_{iS} = 1 & \forall \text{ buyers } i \ (\text{buyer } i \text{ gets exactly 1 set}) \\
& x_{iS} \geq 0 & \forall i, j
\end{aligned}$$

Let $S_1^*, \ldots, S_n^*$ be the allocation at Walrasian equilibrium and let $\{x_{iS}^*\}$ be the optimal LP solution. Then for all set S and buyer i

$$\begin{aligned}
v_i(S_i^*) - p(S_i^*) &\geq v_i(S) - p(S) \\
v_i(S_i^*) - p(S_i^*) &\geq \sum_S x_{iS}^* \left(v_i(S) - p(S) \right).
\end{aligned}$$

The second inequality follows from the fact that $\sum_S x_{iS}^* = 1$. Summing over all buyers,

$$\sum_i \left(v_i(S_i^*) - p(S_i^*) \right) \geq \sum_i \sum_S x_{iS}^* \left(v_i(S) - p(S) \right).$$

$$(\text{social welfare at equilibrium}) - \sum_j p_j \geq (\text{optimal social welfare}) - \sum_j p_j.$$

■

References

[BBM08] Maria-Florina Balcan, Avrim Blum, and Yishay Mansour, "Item Pricing for Revenue Maximization." *Proc. 9th ACM Conference on Electronic Commerce*, pp. 50-59. 2008.